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Material symmetries of micropolar continua equivalent to lattices

P. Trovalusci*, R. Masiani†

Dipartimento di Ingegneria Strutturale e Geotecnica, Università di Roma "La Sapienza", Via A. Gramsci 53, 00197 Roma, Italy

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Abstract

Some aspects concerning the identification of continuum coarse models from fine discrete models are discussed. The preservation of the mechanical power, in the transition from the microscopic to the macroscopic description, is required. A procedure based on the equivalence of the virtual power provides a natural way to select the continuum satisfying this requirement. Having the advantages of an integral procedure, it gives good results if the coarse model is a multifield continuum with strain fields compatible with those of the fine model. In this situation both models share the same material symmetry group. This is shown with reference to rigid particle systems. In particular, the symmetry group of the discrete material is defined and its transformation into that of an equivalent micropolar continuum is studied in detail. Numerical analyses are also performed to investigate the effect of change in the material symmetries.

1. Introduction

According to the terminology adopted by Muncaster (1983), we call 'fine' description a model which provides detailed information about the actual behaviour of a medium. When not all the information provided is essential, it could be convenient to adopt a 'coarse' description retaining only the information sufficient to a global description of the behavior of the material. Techniques focused on the derivation of continuum coarse descriptions from different kinds of fine models have been receiving much attention and have been affording profitable results. Not only in mechanics of solids with microheterogeneities (Weng et al., 1990; Nemat-Nasser and Hori, 1993) but also, although from different points of view, in mechanics of structures when one or two-dimensional models are derived from the more detailed three-dimensional Cauchy model (e.g.

* E-mail: trova@dsg.uniroma1.it
† E-mail: renato@dsg.uniroma1.it

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Ciarlet and Destuynder, 1979; Lembo and Podio-Guidugli, 1991). However, for the former class of problems the transition from the microscale to the macroscale is often problematical because the generally adopted classical continuum cannot be able to account for the complex phenomenology of the original model. In particular it is known that the classical theory fails whenever problems with high stress gradients will be treated.

To overcome the inadequacies of classical coarse models several improvements have been made within the framework of higher order gradient continuum theories (e.g. Bardenhagen and Triantafyllidis, 1994; Mühlhaus, 1995 and the bibliography therein), or within the framework of the micropolar theory (Mühlhaus, 1995; Besdo, 1985; Chang and Ma, 1992; Mühlhaus, 1993; Dai et al., 1990; Dawson and Cundall, 1996). The adoption of a continuum with microstructure[1] is an effective alternative way to retain memory of the fine organization of the material always resorting to the advantages of the macroscopic description. Some recent papers show how to obtain a macroscopic characterization of a fine model by equating the virtual power of its internal actions and the virtual stress power of a suitable coarse model, like a continuum with a fine microstructure (Di Carlo et al., 1990; Masiani et al., 1995; Di Carlo and Nardinocchi, 1995; Mariano and Trovalusci, 1998).[2] This integral method of equivalence has a specific mechanical meaning and does not require, as the asymptotic homogenization techniques, any limit process. Although the question of the methods of identification of equivalent continua is a prominent one, it is also important to decide what type of macroscopic model is to be adopted. In particular we have to decide if a multifield continuum is convenient and which is the right microstructure. We think that the equivalent continuum should at least preserve the mechanical power of the original fine model. The integral procedure mentioned above provides a natural way to select the continuum model which satisfies this requirement. This procedure departs from the molecular theory (e.g. Ericksen, 1977) which deals with the continuum modelling of lattice systems following a kinematical approach. The crucial step of the method lies in establishing a correspondence between the fine and the coarse kinematical descriptors: the deformations of the fine and the coarse models must be compatible in such a way that all the terms of the mechanical power can be uniquely identified and explicitly constitutive functions for the continuum contact actions can be obtained. In other words, since the degrees of freedom of the two models must correspond, once defined the deformations in the original fine model, a continuum with the proper kinematical descriptors must be selected.

As a suitable consequence of the power equivalence, the two models belong to the same class of material symmetry. When the continuum has not the required microstructure, the identification of the overall material properties cannot be done unless to add internal constraints. In this case, the equivalent continuum does not generally preserve the material symmetries of the original unconstrained medium.

[1] As continuum with microstructure we mean, according to Capriz (1989), a body whose deformation is described by more independent field variables. We also termed such a continuum multifield continuum extending the definition to continua with any additional strain field, like higher order gradients, which gives rise to more stress fields than the classical one. On the other hand, it is noticed that microstructured continua with particular internal constraints behave as materials of grade N (Capriz, 1985).

[2] If the internal structure is affine, the material point can be subjected to any homogeneous deformations independent from the macroscopic classical deformation (Capriz, 1989; Grioli, 1990).

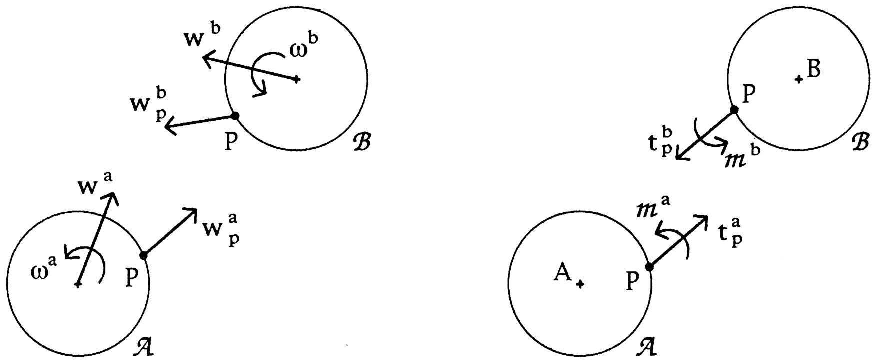

Fig. 1. Discrete model: kinematical (left) and dynamical (right) descriptors.

We exemplify the above using discrete systems made of rigid particles connected by elastic springs. In particular we show that the equivalent continuum must have a rigid local structure; thus the symmetry transformations in the transition from the fine to the coarse description are retained. This result shows its practical relevance if, for example, we consider the block system adopted in (Masiani and Trovalusci, 1996) to model brick masonry-like media. In fact is well known that the response of a blocky material, as well as the response of any other composite material made of a matrix and of a set of inclusions with prominent shape and size, is strongly dependent on the disposition of the blocks. Since the variation of the texture of the blocks implies a change in the classification of the body in terms of material symmetries, it is important to maintain the symmetric transformations of the actual discontinuous material in the coarse model. In the last part of this paper, the effect of change in material symmetries on the response of the discrete material and of the equivalent micropolar continuum is examined on a sample problem with the aid of some numerical analyses; the results are compared with those of the corresponding classical continuum.

2. The rigid particle system

Although the current meaning of fine description is wider, we focused our attention on models represented by a periodic lattice of rigid particles interacting two by two through contact points or surfaces. Examples of such models are brick masonry (Masiani et al., 1995) and jointed rocks (Dai et al., 1990; Dawson and Cundall, 1996).

We adapt the discussion below within the framework of a static linear theory where the components of velocity and angular velocity stand respectively for those of displacement and rotation, while the stress power stands for the internal work. For simplicity the calculations are performed for two-dimensional problems, although there are no theoretical limits to extend the results to the three-dimensional frame.

Choosing an orthonormal base $\{\mathbf{e}_i\}$ $(i = 1, 3)$, we consider two-dimensional systems belonging to the plane defined by $\{\mathbf{e}_1, \mathbf{e}_2\}$ (Fig. 1). The kinematical descriptors are two: the velocity vector $\mathbf{w}^a$ of the center A of a particle $\mathscr{A}$ and the angular velocity of the particle ω^a. As strain measures we assume for each couple of particles, $\mathscr{A}$ and $\mathscr{B}$, the mutual displacement $\mathbf{w}_p$ between two material points $\mathscr{P}^a$ and $\mathscr{P}^b$, both located at P in the reference shape, and the mutual rotation ω_p between the two particles

$$\mathbf{w}_p = \mathbf{w}^b - \mathbf{w}^a + \mathbf{e}_3 \times [\omega^b (P-B) - \omega^a (P-A)],$$
$$\omega_p = (\omega^b - \omega^a) \tag{1}$$

As response functions for the contact actions, the forces $\mathbf{t}_p = \mathbf{t}_p^a = -\mathbf{t}_p^b$ and the couples $m_p = m_p^a = -m_p^b$ (Fig. 1), we consider respectively the linear elastic relations $\mathbf{t}_p = \mathbf{K}_p \mathbf{w}_p$ and $m_p = k_p \omega_p$. The principle of material objectivity demands a restriction on the form of the constitutive functions so that the image of $\mathbf{t}_p$ under a change of frame is $\mathbf{Q}(\mathbf{K}_p \mathbf{w}_p)$, where $\mathbf{Q}$ is an orthogonal tensor.

Chosen a 'module' $\mathscr{M}$—large enough to be a representative part of the discrete system and also sufficiently small to use the localization theorem—the power of the internal actions in $\mathscr{M}$ has the formula

$$\pi(\mathbf{w}_p) = \sum_p (\mathbf{K}_p \mathbf{w}_p \cdot \mathbf{w}_p + k_p \omega_p^2) \tag{2}$$

where the sum on p applies to all the contacts of the module.

As in molecular theory, we select the admissible velocities of the module assuming, in the neighbourhood of a position X, represented by the Euclidean region occupied by $\mathscr{M}$, the affine representation

$$\mathbf{w}^a = \mathbf{w}(X) + \operatorname{grad} \mathbf{w}(X)(A-X)$$
$$\omega^a = \omega(X) + \operatorname{grad} \omega(X) \cdot (A-X) \tag{3}$$

Thus according to Ericksen (1977), being the macroscopic model to be constructed a continuum model accounting for short range interactions, it suffices to consider homogeneous deformations.[3] Therefore, the strain measures (1) can be expressed in terms of the smooth fields[4] $\mathbf{U}(X) = \operatorname{grad} \mathbf{w}(X) - \omega(X)\mathbf{e}_1 \wedge \mathbf{e}_2$ and $\mathbf{u}(X) = \operatorname{grad} \omega(X)$

$$\mathbf{w}_p = \mathbf{U}(X)(B-A) + \mathbf{e}_3 \times [(P-B) \otimes (B-X) - (P-A) \otimes (A-X)]\mathbf{u}(X)$$
$$\omega_p = \mathbf{u}(X) \cdot (B-A). \tag{4}$$

Taking into account eqns (4), the virtual power of the contact actions in $\mathscr{M}$ becomes[5]

[3] As in Bardenhagen and Triantafyllidis (1994), using Taylor series expansion up to the order N, a continuum model of grade N accounting for long range interactions can be constructed.

[4] The skewsymmetric tensor $\omega \mathbf{e}_1 \wedge \mathbf{e}_2 = \omega \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1$ is the tensor corresponding to the axial vector $\omega \mathbf{e}_3$.

[5] From now on the explicit dependence on X can be understood.

$$\pi(\check{\mathbf{U}}, \check{\mathbf{u}}) = \sum_p \{\mathbf{K}_p \mathbf{w}_p \cdot (\check{\mathbf{U}}\mathbf{g}_p + \mathbf{e}_3 \times \mathbf{G}'_p \check{\mathbf{u}}) + k_p \omega_p \check{\mathbf{u}} \cdot \mathbf{g}_p\}$$
$$= \check{\mathbf{U}} \cdot \sum_p (\mathbf{K}_p \mathbf{w}_p \otimes \mathbf{g}_p) + \check{\mathbf{u}} \cdot \sum_p (\mathbf{G}''_p \mathbf{K}_p \mathbf{w}_p + k_p \omega_p \mathbf{g}_p) \tag{5}$$

with

$$\mathbf{g}_p = (B-A)$$
$$\mathbf{G}'_p = [(P-B) \otimes (B-X) - (P-A) \otimes (A-X)]$$
$$\mathbf{G}''_p = (B-X) \otimes \mathbf{e}_3 \times (P-B) - (A-X) \otimes \mathbf{e}_3 \times (P-A)$$

where the sum is extended to each contact of $\mathscr{M}$, and where the symbol '$\check{}$' stands for the adjective 'virtual'. Thence, identifying again the actual strains through the eqns (4)

$$\pi(\check{\mathbf{U}}, \check{\mathbf{u}}) = \check{\mathbf{U}} \cdot \sum_p [\mathbf{K}_p(\mathbf{U}\mathbf{g}_p + \mathbf{e}_3 \times \mathbf{G}'_p \mathbf{u}) \otimes \mathbf{g}_p] + \check{\mathbf{u}} \cdot \sum_p [\mathbf{G}''_p \mathbf{K}_p(\mathbf{U}\mathbf{g}_p + \mathbf{e}_3 \times \mathbf{G}'_p \mathbf{u}) + k_p(\mathbf{g}_p \otimes \mathbf{g}_p)\mathbf{u}]. \tag{6}$$

Finally, it can be shown that π takes the form

$$\pi(\check{\mathbf{U}}, \check{\mathbf{u}}) = \check{\mathbf{U}} \cdot (\mathbb{A}\mathbf{U} + \mathsf{B}\mathbf{u}) + \check{\mathbf{u}} \cdot (\mathsf{C}\mathbf{U} + \mathbf{D}\mathbf{u}), \tag{7}$$

where the constitutive tensors[6] $\mathbb{A} := \mathscr{L}in \to \mathscr{L}in$, $\mathsf{B} := \mathscr{V} \to \mathscr{L}in$, $\mathsf{C} := \mathscr{L}in \to \mathscr{V}$, $\mathbf{D} := \mathscr{V} \to \mathscr{V}$ have components depending on the components of the elastic tensors $\mathbf{K}_p$, on the rotational elasticity constant k_p and on the components of the fabric vectors and tensors $\mathbf{g}_p$, $\mathbf{G}'_p$ and $\mathbf{G}''_p$ of the module. The expressions of these components can be found in Appendix A.

3. Identification of the equivalent continuum

To derive the constitutive relationships of the macroscopic model we suppose that the virtual power of the internal actions of the module (7) is equal to the virtual stress power spent over a neighborhood of the continuum which occupies the same region as the module.[7] In order to uniquely identify the terms of the power it is now clear that the equivalent continuum must admit the fields $\mathbf{U}(X)$ and $\mathbf{u}(X)$ as strain measures. It can be observed that, in the framework of a linear theory, such a continuum is of the Cosserat type, with the velocity field $\mathbf{w}(X)$ and the additional field of the angular velocity (microspin) $\omega(X)$. Denoted $\mathbf{S}(X)$ the tensor field of the stress and $\mathbf{s}(X)$ vector field of the couple stress, we then have

[6] $\mathscr{L}in$ is the set of the linear transformation of the vector space $\mathscr{V}$ on itself.

[7] This approach is consistent with the energy approach often employed in the above mentioned continuum models based on periodical latices. In the molecular theory the point of view is reversed: the lattice and its strain measures are defined in order to obtain the expected macroscopic model.

$$\pi(\check{\mathbf{U}},\check{\mathbf{u}}) = \int_M (\check{\mathbf{U}}\cdot\mathbf{S}+\check{\mathbf{u}}\cdot\mathbf{s})\,\mathrm{d}A, \tag{8}$$

from which, resorting to the localization at X, it follows the approximate equivalence

$$\frac{1}{A}\pi(\check{\mathbf{U}},\check{\mathbf{u}}) = \check{\mathbf{U}}(X)\cdot\mathbf{S}(X)+\check{\mathbf{u}}(X)\cdot\mathbf{s}(X), \tag{9}$$

where A is the area of $\mathscr{M}$. Thence, using the expression (7), we derive the explicit constitutive functions for the continuum contact actions

$$\mathbf{S} = \frac{1}{A}(\mathbb{A}\mathbf{U}+\mathsf{B}\mathbf{u}),\quad \mathbf{s} = \frac{1}{A}(\mathsf{C}\mathbf{U}+\mathbf{D}\mathbf{u}). \tag{10}$$

Whatever the strain measures of the fine model, approximated by series of smooth terms, are, the power equivalence requires a continuum with the proper strain fields.[8] Otherwise, specific constitutive prescriptions must be introduced in the discrete model. For example it is easy to verify that a Cauchy material cannot meet all the requirements of the discrete system described above. The equivalence in terms of density power[9]

$$\frac{1}{A}\pi(\check{\mathbf{U}},\check{\mathbf{u}}) = \mathrm{sym}\,\check{\mathbf{U}}(X)\cdot\mathbf{S}(X) \tag{11}$$

gives the stress tensor $\mathbf{S} = \mathrm{sym}(\mathbb{A}\mathbf{U}+\mathsf{B}\mathbf{u})$ if $\mathrm{skw}\,\check{\mathbf{U}} = 0$ and $\check{\mathbf{u}} = 0$, that is when the microspin equals the macrospin

$$\check{\omega}\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathrm{skw}\,\mathrm{grad}\,\check{\mathbf{w}}, \tag{12}$$

or if

$$\mathrm{skw}(\mathbb{A}\mathbf{U}+\mathsf{B}\mathbf{u}) = \mathbf{0},\quad \mathsf{C}\mathbf{U}+\mathbf{D}\mathbf{u} = \mathbf{0}, \tag{13}$$

When the second of the above equations is verified, the first one represents the balance of angular momentum with zero mass couples. In terms of components it can be written[10]

$$\begin{aligned}&[(\mathbb{A})_{1211}-(\mathbb{A})_{2111}](\mathrm{sym}\,\mathbf{U})_{11}+[(\mathbb{A})_{1222}-(\mathbb{A})_{2122}](\mathrm{sym}\,\mathbf{U})_{22}\\&\quad+[(\mathbb{A})_{1212}-(\mathbb{A})_{2121}+(\mathbb{A})_{1221}-(\mathbb{A})_{2112}](\mathrm{sym}\,\mathbf{U})_{12}\\&\quad-[(\mathbb{A})_{1212}+(\mathbb{A})_{2121}-(\mathbb{A})_{1221}-(\mathbb{A})_{2112}](\theta-\omega) = 0,\end{aligned} \tag{14}$$

with $\theta = \frac{1}{2}(\mathrm{curl}\,\mathbf{w})\cdot\mathbf{e}_3$.

The positions (12) and (13) demand the following restrictions on the original discrete system. Having the internal constraint (12), the velocities that the assembly may undergo must be

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[8] Another way to derive continua having the same power of lattices is shown in (Dawson, 1995) where the kinematical correspondence is obtained by discretizing the continuum displacement fields.

[9] The operators sym and skw give respectively the symmetric and the skewsymmetric part of a second order tensor.

[10] We use the parentheses to represent vectors and tensors in terms of components.

$$\mathbf{w}^a = \mathbf{w}(X) + \operatorname{grad}\mathbf{w}(X)(A-X)$$
$$\omega^a = \theta(X), \tag{15}$$

and the ordinary stress tensor can be identified. With the constitutive prescriptions (13) it has $(\mathbb{A})_{ijhk} = (\mathbb{A})_{jihk}$, $(\mathsf{B})_{ijh} = (\mathsf{B})_{jih}$, $\mathbf{D} = \mathbf{0}$, and then, if we consider hyperelastic materials having the major symmetries, it has

$$(\mathbb{A})_{ijhk} = (\mathbb{A})_{hkij} = (\mathbb{A})_{jihk}, \quad (\mathsf{B})_{ijh} = (\mathsf{C})_{hij} = 0, \quad \mathbf{D} = \mathbf{0}. \tag{16}$$

If the geometrical and the mechanical features are such that the above constraints are satisfied, we can identify a continuum which spends zero power with any skw grad $\mathbf{w}$ and with any $\mathbf{u}$. It can be noticed (see Section 4.1) that two-dimensional periodical assemblies are centrosymmetric materials with $\mathsf{B} = 0$ and $\mathsf{C} = 0$. Moreover, if the particles are very small with respect to the dimension of the body we have $\mathbf{D} = \mathbf{0}$ (Masiani and Trovalusci, 1996). However, if we consider modules having joints between blocks with the same specific material constants, the term $(\mathbb{A})_{1221}$ is always zero and the first constraint in (16) cannot be satisfied by a positive definite constitutive tensor. It follows that a symmetric stress tensor can be obtained when the two shear moduli $(\mathbb{A})_{1212}$ and $(\mathbb{A})_{2121}$ coincide and when the kinematical constraint (12) is satisfied.[11] Therefore, the equivalent classical continuum can be identified if the mutual rotations of the blocks are not considered.

4. Symmetric transformations of discrete and continuum model

To define the symmetry transformations of a lattice system, the mechanical properties of the links as well as the geometry of the assembly must be taken into account. We consider as symmetry transformation an orthogonal transformation $\mathbf{Q}$ of the reference configuration of the module which leaves unchanged the power of the interactions in $\mathscr{M}$[12]

$$\sum_p \{\mathbf{K}_p\mathbf{w}_p \cdot \mathbf{w}_p + k_p\omega_p^2\} = \sum_p \{\mathbf{Q}^T\mathbf{K}_p\mathbf{Q}\tilde{\mathbf{w}}_p \cdot \tilde{\mathbf{w}}_p \cdot k_p\tilde{\omega}_p^2\} \tag{17}$$

with

$$\tilde{\mathbf{w}}_p = \mathbf{w}^b - \mathbf{w}^a + \mathbf{e}_3 \times [\omega^b\mathbf{Q}^T(P-B) - \omega^a\mathbf{Q}^T(P-A)]. \tag{18}$$

It is worth pointing out that the symmetry properties of a material are characterized by the response of the body to the admissible deformations. Hence, the material symmetry group $\mathscr{G}$ can be defined when the class of these deformations has been selected. As we have defined the admissible

[11] This result has already been noticed for the particular case of diagonal constitutive tensors (Masiani and Trovalusci, 1996).

[12] The concept of symmetry group is related to an overall response parameter—the mechanical power—of the module. This is consistent with the Noll's definition of the isotropy group of simple solid materials (Noll, 1958).

velocity fields for the lattice through eqns (3), the definition of $\mathscr{G}$ must be referred to the virtual power $\pi(\check{\mathbf{U}}, \check{\mathbf{u}})$.[13]

An orthogonal transformation $\mathbf{Q}$ acts on the constitutive and the geometric tensors and vectors in such a way that

$$\begin{aligned} \tilde{\mathbf{w}}_p &= \mathbf{U}\mathbf{Q}^T\mathbf{g}_p + \mathbf{e}_3 \times (\mathbf{Q}^T\mathbf{G}'\mathbf{Q})\mathbf{u} \\ \tilde{\omega}_p &= \mathbf{u} \cdot \mathbf{Q}^T\mathbf{g}_p, \end{aligned} \tag{19}$$

and the virtual power of the contact actions of the rotated material becomes

$$\tilde{\pi}(\check{\mathbf{U}}, \check{\mathbf{u}}) = \sum_p \{\mathbf{Q}^T\mathbf{K}_p\mathbf{Q}\tilde{\mathbf{w}}_p \cdot [\check{\mathbf{U}}\mathbf{Q}^T\mathbf{g}_p + \mathbf{Q}^T\mathbf{e}_3 \times (\mathbf{G}'_p\mathbf{Q}\check{\mathbf{u}})] + k_p\tilde{\omega}_p\mathbf{Q}\check{\mathbf{u}} \cdot \mathbf{g}_p\} \tag{20}$$

Alternatively, considering a change of frame that leaves $\tilde{\pi}$ invariant, we can write

$$\begin{aligned} \tilde{\pi}(\check{\mathbf{U}}, \check{\mathbf{u}}) &= \sum_p \{\mathbf{K}_p\mathbf{Q}\tilde{\mathbf{w}}_p \cdot [\mathbf{Q}\check{\mathbf{U}}\mathbf{Q}^T\mathbf{g}_p + \mathbf{e}_3 \times \mathbf{G}'_p\mathbf{Q}\check{\mathbf{u}}] + k_p\tilde{\omega}_p\mathbf{Q}\check{\mathbf{u}} \cdot \mathbf{g}_p\} \\ &= \mathbf{Q}\check{\mathbf{U}}\mathbf{Q}^T \cdot \sum_p [\mathbf{K}_p\mathbf{Q}\tilde{\mathbf{w}}_p \otimes \mathbf{g}_p] + \mathbf{Q}\check{\mathbf{u}} \cdot \sum_p [\mathbf{G}''_p\mathbf{K}_p\mathbf{Q}\tilde{\mathbf{w}}_p + k_p\tilde{\omega}_p\mathbf{g}_p] \\ &= \mathbf{Q}\check{\mathbf{U}}\mathbf{Q}^T \cdot \sum_p [\mathbf{K}_p(\mathbf{Q}\mathbf{U}\mathbf{Q}^T\mathbf{g}_p + \mathbf{e}_3 \times \mathbf{G}'_p\mathbf{Q}\mathbf{u}) \otimes \mathbf{g}_p] \\ &\quad + \mathbf{Q}\check{\mathbf{u}} \cdot \sum_p \{\mathbf{G}''_p\mathbf{K}_p(\mathbf{Q}\mathbf{U}\mathbf{Q}^T\mathbf{g}_p + \mathbf{G}'_p\mathbf{Q}\mathbf{u}) + k_p(\mathbf{g}_p \otimes \mathbf{g}_p)\mathbf{Q}\mathbf{u}\} \\ &= \pi(\mathbf{Q} * \check{\mathbf{U}}, \mathbf{Q} * \check{\mathbf{u}}). \end{aligned}$$

As in the case of eqn (6) it follows[14]

$$\pi(\mathbf{Q} * \check{\mathbf{U}}, \mathbf{Q} * \check{\mathbf{u}}) = \mathbf{Q}\check{\mathbf{U}}\mathbf{Q}^T \cdot (\mathbb{A}\mathbf{Q}\mathbf{U}\mathbf{Q}^T + \mathsf{B}\mathbf{Q}\mathbf{u}) + \mathbf{Q}\check{\mathbf{u}} \cdot (\mathsf{C}\mathbf{Q}\mathbf{U}\mathbf{Q}^T + \mathbf{D}\mathbf{Q}\mathbf{u}). \tag{21}$$

Finally, by equating the virtual power of the interactions before, (7), and after, (21), the action of $\mathbf{Q}$, the condition of invariance of material symmetries is obtained. Considering that $\mathbf{Q}$ preserves the inner product this condition can be written

$$\begin{aligned} &\mathbf{Q}\check{\mathbf{U}}\mathbf{Q}^T \cdot \mathbf{Q}(\mathbb{A}\mathbf{U} + \mathsf{B}\mathbf{u})\mathbf{Q}^T + \mathbf{Q}\check{\mathbf{u}} \cdot \mathbf{Q}(\mathsf{C}\mathbf{U} + \mathbf{D}\mathbf{u}) \\ &\qquad = \mathbf{Q}\check{\mathbf{U}}\mathbf{Q}^T \cdot (\mathbb{A}\mathbf{Q}\mathbf{U}\mathbf{Q}^T + \mathsf{B}\mathbf{Q}\mathbf{u}) + \mathbf{Q}\check{\mathbf{u}} \cdot (\mathsf{C}\mathbf{Q}\mathbf{U}\mathbf{Q}^T + \mathbf{D}\mathbf{Q}\mathbf{u}). \end{aligned} \tag{22}$$

[13] On the other hand, we could consider eqns (3) an internal constraint of the discrete material and define the material symmetry group in such a way that the domain of the power $\pi(\mathbf{w}_p, \omega_p)$ were a subspace whose elements admit the representation (4). Thus, we resort to the concept of local symmetry specified in (Fosdick and Hertog, 1990).

[14] The symbol '$*$' represents the action of an orthogonal transformation $\mathbf{Q}$ on an element of a vector space. In terms of components $(\mathbf{Q} * \mathbb{T})_{ij\ldots m} = (\mathbb{T})_{ab\ldots n}(\mathbf{Q})_{ia}(\mathbf{Q})_{jb\ldots}(\mathbf{Q})_{mn}$.

The set of the orthogonal transformations of the two-dimensional space that verify eqn (22) forms the symmetry group of the discrete material[15]

$$\mathscr{G} = \{\mathbf{Q} \in \mathscr{O}(2) \,|\, \{\mathbf{Q} * (\mathbb{A}\mathbf{U}) = \mathbb{A}(\mathbf{Q} * \mathbf{U}), \forall \mathbf{U} \in \mathscr{L}in\} \cap \{\mathbf{Q} * (\mathsf{B}\mathbf{u}) = \mathsf{B}(\mathbf{Q} * \mathbf{u}), \forall \mathbf{u} \in \mathscr{V}\} \cap \{\mathbf{Q} * (\mathsf{C}\mathbf{U}) = \mathsf{C}(\mathbf{Q} * \mathbf{U}), \forall \mathbf{U} \in \mathscr{L}in\} \cap \{\mathbf{Q} * (\mathbf{D}\mathbf{u}) = \mathbf{D}(\mathbf{Q} * \mathbf{u}), \forall \mathbf{u} \in \mathscr{V}\}\}. \quad (23)$$

As identifying all the terms of the density stress power the constitutive relations (10) are derived, it follows that the set (23) also defines the material symmetry group of the micropolar equivalent continuum. Consequently, the symmetry group of the Cauchy continuum equivalent to a discrete assembly of particles which cannot rotate independently of one another, $\mathscr{G}_c = \{\mathbf{Q} \in \mathscr{O}(2) \,|\, \mathbf{Q} * \text{sym}(\mathbb{A}\mathbf{U}) = \text{sym}\,\mathbb{A}(\mathbf{Q} * \mathbf{U}),\ \forall \mathbf{U} \in \mathscr{S}ym\}$,[16] is wider than that of the structured discrete assembly described above.[17]

4.1. Classification of the equivalent material

By defining the elastic constants and the geometry of the discrete system, the power equivalence procedure provides a simple way to identify all the constitutive parameters of the continuum medium. The number of independent parameters to be identified depends on the symmetry class of the material. Employing the definition (23) the problem of classification of the equivalent micropolar material can be undertaken.

If the constitutive tensor $\mathbf{K}_p$ of the joints is symmetric, we identify an hyperelastic material having the major symmetries. That is, said $\mathbf{A}$, $\mathbf{B}$, $\mathbf{a}$, $\mathbf{b}$ elements of a vector spaces, $\mathbb{A}\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbb{A}\mathbf{B}$; $\mathsf{B}\mathbf{a} \cdot \mathbf{A} = \mathbf{a} \cdot \mathsf{C}\mathbf{A}$; $\mathbf{D}\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{D}\mathbf{b}$. It follows that an anisotropic medium with micropolar structure has, in the two-dimensional case, twenty-one independent elasticities.

In order to classify the response of the structured continuum within the ordinary material symmetry class, we analyse the effect of some finite subgroups of $\mathscr{O}(2)$ on the components of its constitutive tensors.[18] Note that in classical elasticity, being the constitutive tensor an even order tensor, the material symmetry groups contain both the identity transformation $\mathbf{I}$ and the inverse transformation $-\mathbf{I}$. Hence, a group containing a proper orthogonal transformation $\mathbf{Q}$ also contains the corresponding improper transformation $-\mathbf{Q}$ and it suffices to consider only the subgroups of $\mathscr{S}\mathscr{O}(2)$.[19] If a constitutive tensor is of odd order, like B and C, we have $-\mathbf{I} * \mathsf{B}\mathbf{a} \neq \mathsf{B}(-\mathbf{I} * \mathbf{a})$. Then, if $\mathbf{Q}$ belongs to the symmetry group, this does not mean that $-\mathbf{Q}$ also belongs to this group. Therefore, the full group $\mathscr{O}(2)$ must be considered.

Dealing with macroscopic models based on lattice systems, we thought natural to consider only

[15] Being the material here treated simple solids, the symmetry group is a subgroup of the full orthogonal group in two dimensions $\mathscr{O}(2)$.

[16] $\mathscr{S}ym$ is the subset of the symmetric transformations of $\mathscr{L}in$.

[17] As the microstructure is the orientation of the particles, $\mathscr{G}$ and $\mathscr{G}_c$ are the material symmetry group in the micro and in the macro domain respectively. See (Ilcewicz et al., 1965).

[18] Recently, Xiao (1995) has systematized the characteristic representations of constitutive tensors in terms of invariant quantities for each crystallographic class in classical and in micropolar elasticity.

[19] $\mathscr{S}\mathscr{O}(2)$ is the group of the proper orthogonal transformations of the two-dimensional space.

Table 1
Constitutive prescriptions due to the action of the considered groups

$\mathscr{D}_4$, Orthotetragonal		
$\mathscr{D}_2$, Orthotropic		
$\mathscr{Z}_2$, Centrosymmetric		
$\mathsf{B} = 0$	$(\mathbb{A})_{1112} = 0$	$(\mathbb{A})_{1111} = (\mathbb{A})_{2222}$
$\mathsf{C} = 0$	$(\mathbb{A})_{1121} = 0$	$(\mathbb{A})_{1212} = (\mathbb{A})_{2121}$
	$(\mathbb{A})_{2212} = 0$	$(\mathbf{D})_{11} = (\mathbf{D})_{22}$
	$(\mathbb{A})_{2221} = 0$	
	$(\mathbf{D})_{12} = 0$	

the finite subgroups of $\mathscr{O}(2)$ that are symmetry groups of congruent polygons.[20] We consider the cyclic group of order two $\mathscr{Z}_2$, the dihedral group of order four $\mathscr{D}_2$, and the dihedral group of order eight $\mathscr{D}_4$, with $\mathscr{Z}_2 \in \mathscr{D}_2 \in \mathscr{D}_4$ defined in Appendix B. These subgroups correspond respectively to the symmetry groups of a parallelogram, a rectangle and a square. The action of these groups on the elastic tensors of a material defines the more usual class of anisotropy of material bodies.

The constitutive prescriptions that the action of the three groups implies on the elastic tensors of the structured material are summarized in Table 1.

The material whose response is invariant under $\mathscr{Z}_2$ belongs to the class of the centrosymmetric materials. In this case the number of the independent elastic constants becomes thirteen. It can be observed that for periodic lattices the module occupies a ‘fundamental region’ whose translation must cover completely the space. Therefore it must have the particles and the mechanical properties of the links distributed according to the central symmetry. Consequently, the response of a material equivalent to periodic assemblies does not change under the action of $\mathscr{Z}_2$. A material with symmetry group $\mathscr{D}_2$ is an orthotropic material with eight independent elasticities. Finally, the material with symmetry group $\mathscr{D}_4$ belongs to the class of orthotetragonal materials and the independent elasticities are five.

5. Block masonry. Some numerical results

To explain the consequences of the change in the material symmetries on the response of the body we analyse, as sample problem, the behaviour of masonry-like systems. Masonry walls are constituted of bricks or blocks arranged in various regular dispositions. As noticed above, a system of rigid elements arranged according to a periodical texture belongs at least to the class of centrosymmetric materials. By varying the disposition of the blocks according to the ordinary

[20] The isometries of a figure of the Euclidean space form the group of symmetries of the figure.

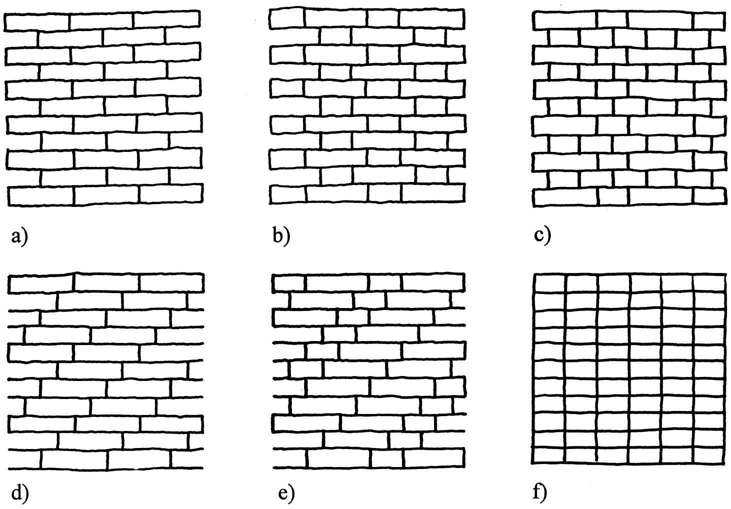

Fig. 2. Examples of walls made of different block textures. The masonries (a), (b) and (c) are orthotropic; (d) and (e) are centrosymmetric; (f) is orthotetragonal.

masonry textures (Fig. 2), we obtain centrosymmetric, orthotropic and orthotetragonal materials. The response of the equivalent Cosserat material is then invariant respectively under the action of the groups $\mathscr{Z}_2$, $\mathscr{D}_2$ and $\mathscr{D}_4$ considered in the previous section.

In earlier attempts the effectiveness of the described identification procedure with reference to block masonry systems has been tested (Masiani and Trovalusci, 1996; Masiani and Trovalusci, 1995). In particular some analyses have been performed on walls modeled as discrete systems, as Cosserat as well as Cauchy equivalent continuum. This comparison showed that the micropolar model is preferable because the strain and the stress tensors are unsymmetric and because it accounts for the blocks relative rotations and the size effects.

Here a further distinctive feature of the micropolar continuum with respect to the classical one is pointed out; that is the retention of the material symmetries of the discrete model. The sample analysed is a 800×800 panel subjected to a pair of contact forces acting along the diagonal. Two different block textures are considered, of the kind (a) and (d) in Fig. 2. The blocks have dimensions 20×80. The elastic tensor for the joint in the discrete model is assumed to be[21]

$$(\mathbf{K}_p) = \begin{pmatrix} 25 \cdot 10^3 & 0 \\ 0 & 5 \cdot 10^3 \end{pmatrix}$$

while the constant k_p is computed from the width of the joint b as: $k_p = (\mathbf{K}_p)_{11}(b/2)^2$. Firstly, the two discrete problems have been solved by modelling the bricks with nodes related by rigid constraint equations. The results are compared to those of the Cosserat and the Cauchy equivalent

[21] In a local frame, with $\mathbf{e}_1$ normal to the joint surface.

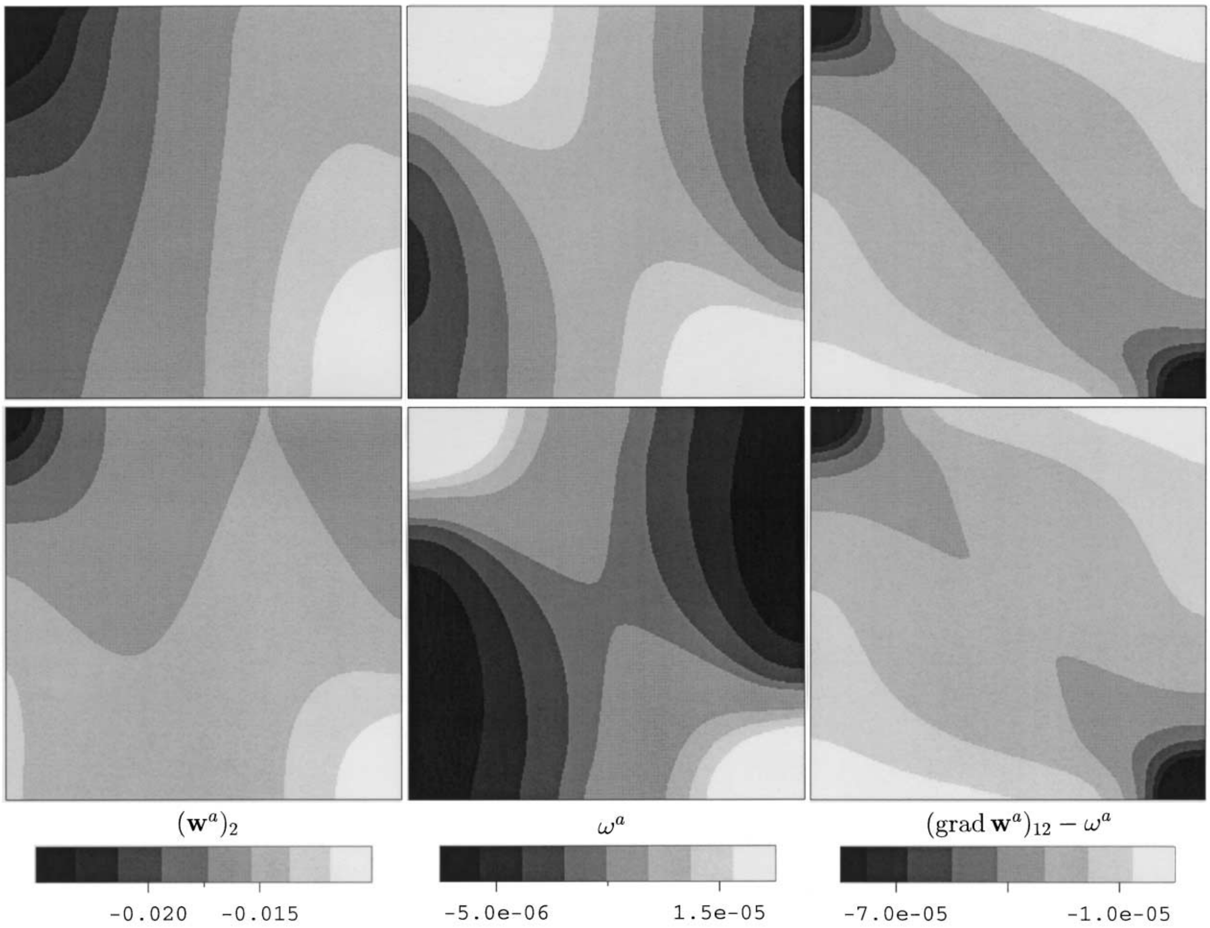

Fig. 3. Sample problem with texture (a), top, and (d), bottom. Discrete material : contour lines of the vertical displacement and the rotation of the blocks, and of the term corresponding to the strain $(\mathbf{U})_{12}$.

continuum, derived as in (Masiani and Trovalusci, 1996) using a finite elements procedure. A membrane element with in-plane rotation has been performed to solve the Cosserat problem. The elasticities of the equivalent continua are evaluated using the expression in Appendix A and are listed in Table 2. The 1-axis is assumed to be horizontal, parallel to the bed joints. Note, in the Cosserat materials, the considerable difference between the terms $\mathbb{A}_{1212}$ and $\mathbb{A}_{2121}$ relative to the shear stresses tangential and normal to the bed joints.

Figures 3–5 show the contour lines of some components of the solution, respectively for the discrete, the Cosserat and the Cauchy models. The micropolar solution agrees very well with the discrete one. Besides, the response of the discrete and of the micropolar continuum, centrosymmetric in the first case and orthotropic in the second one, appears strongly influenced by the arrangement of the blocks, while the response of the classical continuum, which in both cases is an orthotropic material, does not vary considerably.

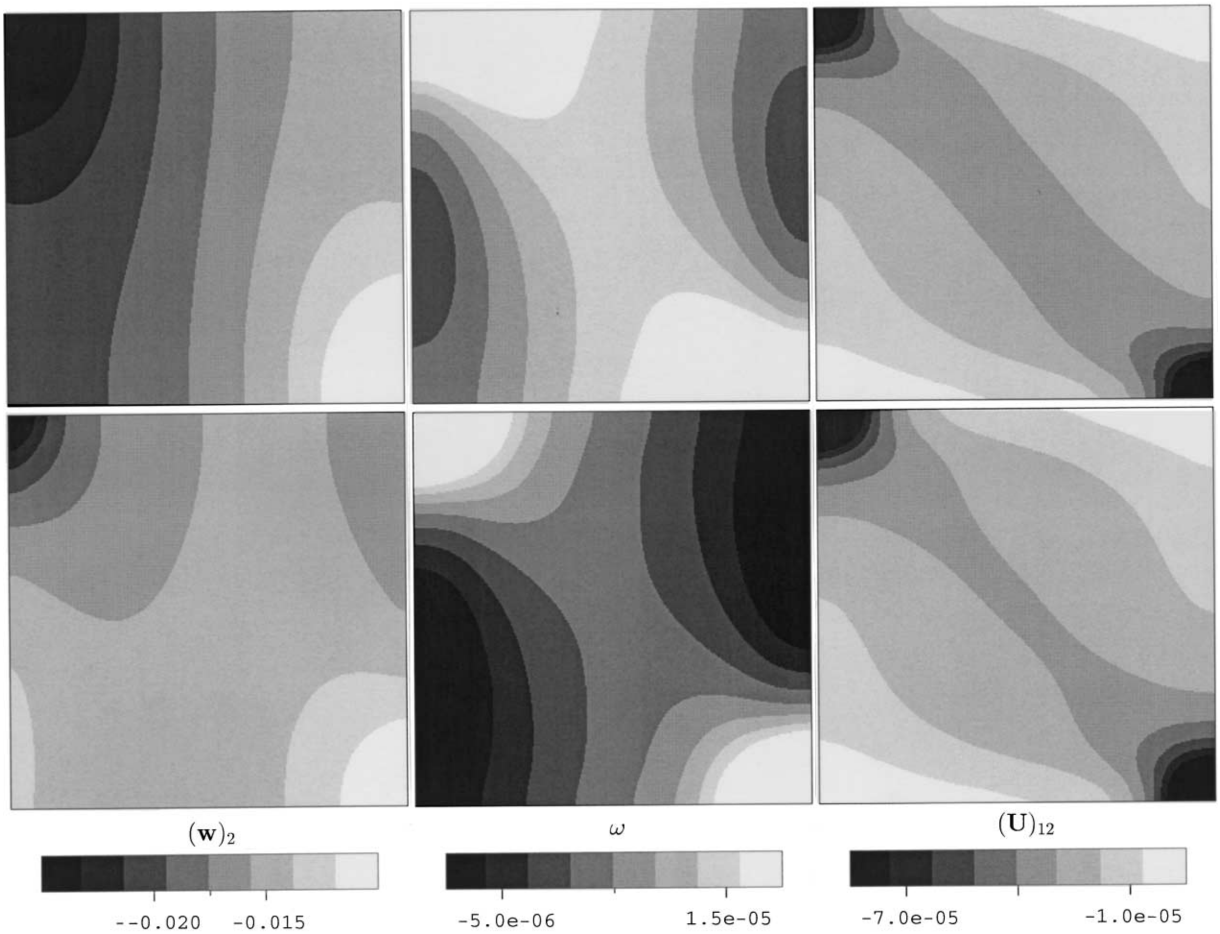

Fig. 4. Sample problem with texture (a), top, and (d), bottom. Cosserat material: contour lines of the vertical displacement, the microrotation and one of the shearing strains.

6. Conclusions

While the usefulness of continuum modelling of discrete systems is evident, its effectiveness is influenced by the nature of the macroscopic model adopted. The kinematical and dynamical descriptors of a Cauchy continuum are often not sufficient and models with more fields of strain and stress must be adopted. To choose the continuum model we assume that the mechanical powers of the two models coincide for any corresponding motions. In this situation the material symmetries in the transition from the microscopic to the macroscopic description are preserved. The effect of the correspondence between the symmetry groups is exemplified by a masonry wall made of blocks with different textures.

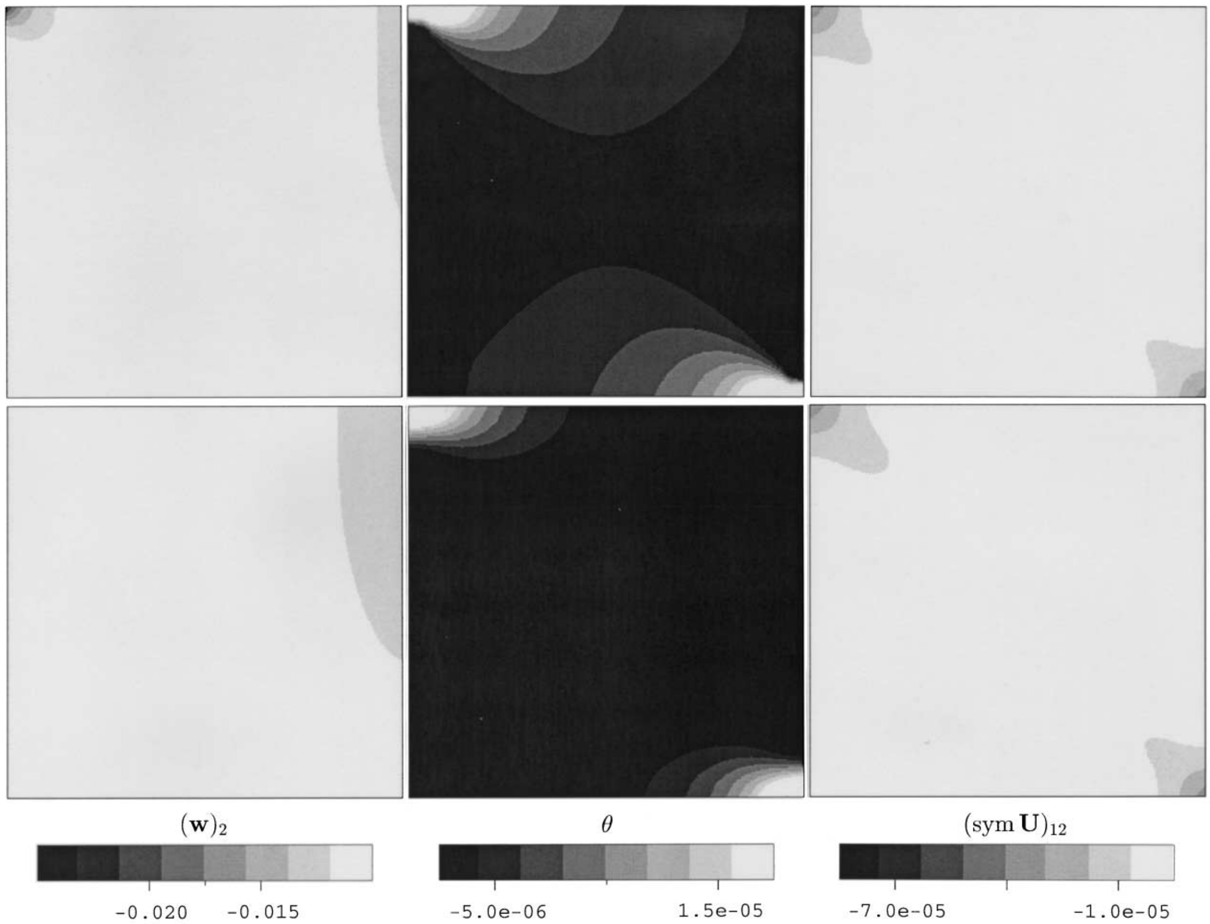

Fig. 5. Sample problem with texture (a), top, and (d), bottom. Cauchy material : contour lines of the vertical displacement, the local rigid rotation and the shearing strain.

Table 2
Continuum model elasticities

	Cosserat elasticities		Cauchy elasticities	
	Texture (a)	Texture (d)	Texture (a)	Texture (d)
$(\mathbb{A})_{1111}$	$120 \cdot 10^3$	$115 \cdot 10^3$	$120.0 \cdot 10^3$	$115 \cdot 10^3$
$(\mathbb{A})_{2222}$	$25 \cdot 10^3$	$25 \cdot 10^3$	$25.0 \cdot 10^3$	$25 \cdot 10^3$
$(\mathbb{A})_{1212}$	$5 \cdot 10^3$	$5 \cdot 10^3$	$62.5 \cdot 10^3$	$50 \cdot 10^3$
$(\mathbb{A})_{2121}$	$120 \cdot 10^3$	$95 \cdot 10^3$	$62.5 \cdot 10^3$	$50 \cdot 10^3$
$(\mathbf{D})_{11}$	$500 \cdot 10^5$	$325 \cdot 10^5$		
$(\mathbf{D})_{22}$	$100 \cdot 10^5$	$175 \cdot 10^5$		
$(\mathbf{D})_{12}$	0	$-150 \cdot 10^5$		

Appendix A. Constitutive parameters

The stress–strain relationships (10) in terms of components are[22]

$$(\mathsf{S})_{ij} = \frac{1}{A}[(\mathbb{A})_{ijhk}(\mathbf{U})_{hk} + (\mathsf{B})_{ijh}(\mathbf{u})_h], \quad (\mathbf{s})_i = \frac{1}{A}[(\mathsf{C})_{ijh}(\mathbf{U})_{jh} + (\mathbf{D})_{ij}(\mathbf{u})_j]. \tag{24}$$

We consider hyperelastic materials for which the following symmetries hold: $(\mathbb{A})_{ijhk} = (\mathbb{A})_{hkij}$; $(\mathsf{B})_{ijh} = (\mathsf{C})_{hij}$; $(\mathbf{D})_{ij} = (\mathbf{D})_{ji}$. Moreover, we consider centrosymmetric lattices with the tensors B and C zero. The explicit expressions for the independent components of the elasticity tensors $\mathbb{A}$ and $\mathbf{D}$ are

$$(\mathbb{A})_{1111} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{11}(B-A)_1^2 \quad (\mathbb{A})_{2222} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{22}(B-A)_2^2$$

$$(\mathbb{A})_{1112} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{11}(B-A)_1(B-A)_2 \quad (\mathbb{A})_{1121} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{12}(B-A)_1^2$$

$$(\mathbb{A})_{1122} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{12}(B-A)_1(B-A)_2 \quad (\mathbb{A})_{2212} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{21}(B-A)_2^2$$

$$(\mathbb{A})_{2221} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{22}(B-A)_1(B-A)_2 \quad (\mathbb{A})_{1212} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{11}(B-A)_2^2$$

$$(\mathbb{A})_{1221} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{12}(B-A)_1(B-A)_2 \quad (\mathbb{A})_{2121} = \frac{1}{A}\sum_p(\mathbf{K}_p)_{22}(B-A)_1^2$$

$$\begin{aligned}(\mathbf{D})_{11} = \frac{1}{A}\sum_p \{&(\mathbf{K}_p)_{11}[(B-X)_1^2(P-B)_2^2 + (A-X)_1^2(P-A)_2^2 \\ &-2(B-X)_1(A-X)_1(P-B)_2(P-A)_2] \\ &+(\mathbf{K}_p)_{12}[-(B-X)_1^2(P-B)_1(P-B)_2 - (A-X)_1^2(P-A)_1(P-A)_2 \\ &+(B-X)_1(A-X)_1(P-B)_1(P-A)_2 + (B-X)_1(A-X)_1(P-B)_2(P-A)_1] \\ &+(\mathbf{K}_p)_{21}[-(B-X)_1^2(P-B)_1(P-B)_2 - (A-X)_1^2(P-A)_1(P-A)_2 \\ &+(B-X)_1(A-X)_1(P-B)_2(P-A)_1 + (B-X)_1(A-X)_1(P-B)_1(P-A)_2] \\ &+(\mathbf{K}_p)_{22}[(B-X)_1^2(P-B)_1^2 + (A-X)_1^2(P-A)_1^2 \\ &-2(B-X)_1(A-X)_1(P-B)_1(P-A)_1] \\ &+k_p(B-A)_1^2\}\end{aligned}$$

[22] Let $\{\mathbf{e}_n\}$ $(n = 1,4)$ be an orthonormal base, the components of a tensor are $\mathbf{T}_{ijhk} = \mathbf{T}\cdot\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_h\otimes\mathbf{e}_k$.

$$
\begin{aligned}
(\mathbf{D})_{12} = \frac{1}{A}\sum_p \{ & (\mathbf{K}_p)_{11}[(B-X)_1(B-X)_2(P-B)_2^2+(A-X)_1(A-X)_2(P-A)_2^2 \\
& -(B-X)_2(A-X)_1(P-B)_2(P-A)_2-(B-X)_1(A-X)_2(P-B)_2(P-A)_2] \\
& +(\mathbf{K}_p)_{12}[-(B-X)_1(B-X)_2(P-B)_1(P-B)_2 \\
& -(A-X)_1(A-X)_2(P-A)_1(P-A)_2 \\
& +(B-X)_2(A-X)_1(P-B)_1(P-A)_2+(B-X)_1(A-X)_2(P-B)_2(P-A)_1] \\
& +(\mathbf{K}_p)_{21}[(-(B-X)_1(B-X)_2(P-B)_1(P-B)_2 \\
& -(A-X)_1(A-X)_2(P-A)_1(P-A)_2 \\
& +(B-X)_2(A-X)_1(P-B)_2(P-A)_1+(B-X)_1(A-X)_2(P-B)_1(P-A)_2] \\
& +(\mathbf{K}_p)_{22}[(B-X)_1(B-X)_2(P-B)_1^2+(A-X)_1(A-X)_2(P-A)_1^2 \\
& -(B-X)_2(A-X)_1(P-B)_1(P-A)_1-(B-X)_1(A-X)_2(P-B)_1(P-A)_1] \\
& +k_p(B-A)_1(B-A)_2\}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{D})_{22} = \frac{1}{A}\sum_p \{ & (\mathbf{K}_p)_{11}[(B-X)_2^2(P-B)_2^2+(A-X)_2^2(P-A)_2^2 \\
& -2(B-X)_2(A-X)_2(P-B)_2(P-A)_2] \\
& +(\mathbf{K}_p)_{12}[-(B-X)_2^2(P-B)_1(P-B)_2-(A-X)_2^2(P-A)_1(P-A)_2 \\
& +(B-X)_1(A-X)_1(P-B)_1(P-A)_2+(B-X)_1(A-X)_1(P-B)_2(P-A)_1] \\
& +(\mathbf{K}_p)_{21}[-(B-X)_1^2(P-B)_1(P-B)_2-(A-X)_1^2(P-A)_1(P-A)_2 \\
& +(B-X)_2(A-X)_2(P-B)_2(P-A)_1+(B-X)_2(A-X)_2(P-B)_1(P-A)_2] \\
& +(\mathbf{K}_p)_{22}[(B-X)_2^2(P-B)_1^2+(A-X)_2^2(P-A)_1^2 \\
& -2(B-X)_2(A-X)_2(P-B)_1(P-A)_1] \\
& +k_p(B-A)_2^2\}.
\end{aligned}
$$

Appendix B. Finite subgroups of $\mathcal{O}(2)$

A group can be defined by the generators and their relations of definition. With these relations, the possible compositions of generators can be parted in classes of equivalence represented by the elements of the group. Our attention is focused on the finite subgroup of the orthogonal group $\mathcal{O}(2)$ in the two-dimensional space. A finite subgroup of $\mathcal{O}(2)$ is either a cyclic group of order n, $\mathcal{Z}_n$, associated to the rotations of a regular polygon of n sides, or a dihedral group of order $2n$, $\mathcal{D}_n$, associated both to the rotations and to the reflections of a regular n-gon. (e.g. Grove and Benson, 1985). In particular, the elements of $\mathcal{Z}_n$ are successive applications of the rotation $\mathbf{R}$ through an angle of amplitude $\phi(\mathbf{R}) = 2\pi/n$. The relation of definition of this group is $\mathbf{R}^n = \mathbf{I}$, and the n elements of the group are the identity transformation $\mathbf{I}$ and the rotations $\mathbf{R}, \ldots, \mathbf{R}^{n-1}$. The dihedral groups have two generators which can be a rotation $\mathbf{R}$ and a reflection $\mathbf{F}$.[23] The relations of definition for $\mathscr{D}_n$ are $\mathbf{R}^n = \mathbf{I}$, $\mathbf{F}^2 = \mathbf{I}$, $(\mathbf{FR})^2 = \mathbf{I}$, and the list of its elements is $\mathbf{I}, \mathbf{R}, \ldots, \mathbf{R}^{n-1}, \mathbf{F}, \mathbf{FR}, \ldots, \mathbf{FR}^{n-1}$.

The generators of the subgroups considered in this paper are selected as follows.

The group $\mathscr{Z}_2$ is generated by the rotation $\mathbf{R}_3(\pi)$ through the angle π about the out-of-plane axis $\mathbf{e}_3$, which in two dimensions coincides with the central inversion, $-\mathbf{I}$.

The group $\mathscr{D}_2$ is also called four-group since it also corresponds to the group of the geometrical symmetries of the rectangle. As generators we select the rotation $\mathbf{R}_3(\pi)$ and the reflection with respect to the axis $\mathbf{e}_2$, $\mathbf{F}_1 = \mathbf{I} - 2(\mathbf{e}_1 \otimes \mathbf{e}_1)$.

Finally, the group $\mathscr{D}_4$ is associated to the geometrical symmetries of the square. Its eight elements are generated by the rotation $\mathbf{R}_3(\pi/2) = \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1$ and by the reflection $\mathbf{F}_1$.

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[23] Any reflection of the two-dimensional space can be obtained as a rotation in the three-dimensional space through the angle π about the axis of reflection. Using two elements of the proper orthogonal transformations group $\mathscr{SO}(3)$, the dihedral group of order n in the three-dimensional space, isomorphic to $\mathscr{D}_n \in \mathscr{O}(2)$, can be alternatively generated.

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